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Classical trajectories of 1D complex non-Hermitian Hamiltonian systems

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Abstract
Classical motion of complex 1D non-Hermitian Hamiltonian systems is investigated analytically to identify periodic, unbounded and chaotic trajectories. Expressions for the Lyapunov exponent for 1D complex Hamiltonians are derived. Complex potentials $V_1(x) = \frac{1}{2}\mu x^2$ and $V_2(x) = \mu x^3$ are studied in detail and their Lyapunov exponents are obtained analytically. It was found that when μ is complex all the trajectories of V_1 are chaotic with Lyapunov exponent $|\mathrm{Im}(\mu)|$ and most of the trajectories of V_2 are periodic when μ is pure imaginary. But for other complex values of μ trajectories of V_2 are non-periodic and show infinite oscillations. Unbounded neighbouring trajectories of V_2 show power-law divergence rather than exponential divergence as in the case of V_1.

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1. Introduction

Recently, non-Hermitian Hamiltonians have attracted much interest. Particularly, PT symmetric Hamiltonians received special attention due to the fact that their spectra are entirely real as long as PT symmetry is not spontaneously broken [1–10]. Recently, Mostafazadeh [11–13] has generalized PT symmetry by pseudo Hermiticity and showed that quantum energy eigenvalues of such systems are either real or they come as conjugate pairs. A Hamiltonian is said to be η-pseudo-Hermitian if $H^\dagger = \eta H \eta^{-1}$, where $\dagger$ denotes their adjoint operator and η is a Hermitian invertible linear operator. Several recent investigations have been carried out on 1D and 2D pseudo-Hermitian systems [14–17] quantum mechanically.

In ordinary quantum mechanics (Hamiltonians are Hermitian), it is usual to replace classical position x and momentum p in the classical Hamiltonians by corresponding quantum mechanical operators $\hat{x}$ and $\hat{p}$ to obtain quantum Hamiltonians. Conversely, by replacing $\hat{x}$ and $\hat{p}$ for position x and momentum$p$ in quantum Hamiltonians, one can obtain classical Hamiltonians. It is not clear whether, in general, pseudo-Hermitian quantum Hamiltonians admit classical counterparts as such. However, recent studies on certain 1D and 2D systems have shown a correspondence between quantum and classical mechanics of pseudo-Hermitian systems. The quantum energy eigenvalues can be obtained by a higher order version of the Bohr–Sommerfeld-type quantization rule which uses classical momentum functions as in the case of real Hermitian systems [18]. Further, the zeros of the quantum mechanical wavefunctions of PT-symmetric Hamiltonians can also be predicted semiclassically by using generalized classical moments [19]. Certain semiclassical methods which were developed for quantizing multidimensional real Hermitian Hamiltonians can be applied to 2D PT-symmetric complex systems to obtain quantum mechanical energies as well. Further, quantum mechanical frequency spectra of certain pseudo-Hermitian Hamiltonians can be predicted with their classical trajectories [20]. Therefore, at least, for some non-Hermitian systems, a classical and quantum correspondence exists.

Recently, Bender *et al* [21] have studied the classical motion of a class of PT-symmetric Hamiltonian system $H = p^2 + x^2(\mathrm{i}x)^\epsilon$ in great detail. They found that this system exhibits two phases. During the first phase, $\epsilon \geqslant 0$ and the energy spectrum of the above Hamiltonian is real and positive due to PT symmetry. However, during the second phase, when $-1 < \epsilon < 0$, the spectrum contains an infinite number of complex eigenvalues and a finite number of real eigenvalues because PT symmetry is spontaneously broken. They found that this phase transition which occurs at $\epsilon = 0$ manifests itself in both the quantum mechanical system and the corresponding classical system. In this paper we are mainly concerned with pseudo-Hermitian extension of PT symmetric systems. It would be interesting to find out how such 1D pseudo-Hermitian systems behave classically and whether these systems contain irregular/chaotic trajectories. It is also important to investigate the sensitivity of the trajectories to initial conditions. This is usually achieved by calculating Lyapunov exponents. For 1D real conservative systems, the entire motion is regular. This is due to the fact that for such systems energy E is the only possible constant of motion and it forces the trajectory to be regular. However, 1D complex systems can have both regular and irregular motion.

In this paper we investigate two types of 1D complex non-Hermitian Hamiltonian systems classically. One type consists of 1D PT symmetric systems with real energy spectra and the other type contains 1D complex non-pseudo Hermitian Hamiltonians with entire spectra complex. However, one can construct separable 2D pseudo-Hermitian Hamiltonian systems from such 1D complex non-pseudo Hermitian Hamiltonians (type 2) such that quantum energy eigenvalues are either real or they come as conjugate pairs. Such a 2D pseudo-Hermitian Hamiltonian is of the form

$$H(p_x, p_y, x, y) = \frac{p_x^2}{2} + \frac{p_y^2}{2} + V(x) + V^*(y) \tag{1}$$

where $*$ represents complex conjugation and the 1D non-pseudo Hermitian Hamiltonian from which this 2D Hamiltonian is constructed is

$$H(p_x, x) = \frac{p_x^2}{2} + V(x). \tag{2}$$

Note that $H(p_x, p_y, x, y)$ is pseudo-Hermitian with respect to the exchange operator $(\eta : p_x \longleftrightarrow p_y, x \longleftrightarrow y)$. Therefore both 1D Hamiltonian types mentioned above are relevant in the context of pseudo Hermiticity. In this paper, we address the issue of whether there is a correlation between classically regular/periodic motion and the reality of the quantum eigenvalues for 1D pseudo-Hermitian Hamiltonian systems. We do this by analysing some 1D pseudo-Hermitian Hamiltonians analytically.

The outline of this paper is organized as follows. In section 2, a general expression for Lyapunov exponent for complex 1D Hamiltonian systems is derived. We investigate the classical motion of a complex harmonic oscillator and calculate the Lyapunov exponent analytically in section 3. In section 4, first we solve the classical equation of motion for the system $H(p_x, x) = \frac{p_x^2}{2} + \mu x^3$, where μ is a complex number. Finally, we study in detail trajectories of this system in complex phase space and show that neighbouring non-periodic trajectories have power-law divergence rather than exponential divergence as in the case of a complex harmonic oscillator.

2. Lyapunov exponent for complex 1D Hamiltonian systems

In this section, an expression for Lyapunov exponents is derived for 1D complex Hamiltonians. We assume that the Hamiltonian is of the form

$$H = \frac{p^2}{2} + V(x) \tag{3}$$

where $V(x)$ is a complex potential. The classical equation of motion for this Hamiltonian is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = p = \sqrt{E - V(x)}. \tag{4}$$

By rearranging and integrating (4) we have

$$\mathcal{F}(x, E) = \int \frac{\mathrm{d}x}{\sqrt{E - V(x)}} = t + \alpha \tag{5}$$

where α is an arbitrary constant which is determined by initial conditions. Note also that both time t in the right-hand side of (5) and the energy E are assumed to be real. Now assume that it is possible to invert equation (5) and

$$x(t, \alpha) = G(t + \alpha, E) \tag{6}$$

where $G(t + \alpha, E)$ is the inverse of the function $\mathcal{F}(x, E)$. Since $V(x)$ is a complex function we solve the classical equations of motion in the complex plane. Therefore phase space is complex with two phase space variables x and p. The arbitrary constant α determines the value of x at time $t = 0$ and hence

$$\alpha = \mathcal{F}(x(0), E). \tag{7}$$

Note that for various values of α we have trajectories starting with various initial conditions. $p(0, \alpha)$ is automatically fixed when the total energy of the system E is specified,

$$p(0, \alpha) = \sqrt{E - V(x(0, \alpha))}. \tag{8}$$

In order to calculate the Lyapunov exponent for this system, we monitor the time evolution of two neighbouring trajectories which are infinitesimally separated at $t = 0$. Consider two such trajectories $x(t, \alpha)$ and $x(t, \alpha + \delta\alpha)$. Using the definitions given in [22, 23], we obtain Lyapunov exponents for 1D systems as

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \ln\left[\left|\frac{\Delta x(t)}{\Delta x(0)}\right| + \left|\frac{\Delta p(t)}{\Delta p(0)}\right|\right] \tag{9}$$

where $\Delta x(t)$ and $\Delta p(t)$ are given by

$$\Delta x(t) = x(t, \alpha + \delta\alpha) - x(t, \alpha) \tag{10}$$

$$\Delta p(t) = p(t, \alpha + \delta\alpha) - p(t, \alpha) \tag{11}$$

respectively and $\|$ denotes the complex norm.

By linearizing the above equations [23] we have

$$\Delta x(t) = \frac{\partial x(t, \alpha)}{\partial \alpha} \Delta\alpha. \tag{12}$$

Therefore

$$\Delta x(t) = \frac{\partial G}{\partial \alpha} \Delta \alpha \tag{13}$$

$$\frac{\Delta x(t)}{\Delta x(0)} = \frac{\frac{\partial G}{\partial \alpha}}{\frac{\partial G}{\partial \alpha}(t = 0)}. \tag{14}$$

Similarly, using the relation

$$p(t, \alpha) = \frac{\partial G(t, \alpha, E)}{\partial t} \tag{15}$$

we obtain $\frac{\Delta p(t)}{\Delta p(0)}$ as

$$\frac{\Delta p(t)}{\Delta p(0)} = \frac{\frac{\partial^2 G}{\partial t \partial \alpha}}{\frac{\partial^2 G}{\partial t \partial \alpha}(t = 0)}. \tag{16}$$

Now we write Lyapunov exponent for the system in (3) as

$$\lambda = \lim_{t \to \infty} \frac{1}{t} \ln \left[\left| \frac{\frac{\partial G}{\partial \alpha}}{\frac{\partial G}{\partial \alpha}(t = 0)} \right| + \left| \frac{\frac{\partial^2 G}{\partial t \partial \alpha}}{\frac{\partial^2 G}{\partial t \partial \alpha}(t = 0)} \right| \right]. \tag{17}$$

This is the Lyapunov exponent for 1D systems in terms of function G.

3. Classical motion and Lyapunov exponents of the complex harmonic oscillator

First we investigate the classical motion of the harmonic oscillator with complex frequencies. The Hamiltonian of such systems is written as

$$H = \frac{p^2}{2} + \frac{1}{2}\mu^2 x^2 \tag{18}$$

where $\mu = \mu_r + \mathrm{i}\mu_i$ is a complex quantity. Note that this system has two turning points x_1 and x_2 given by

$$x_1 = \frac{\sqrt{2E}}{\mu} \tag{19}$$

$$x_2 = -\frac{\sqrt{2E}}{\mu}. \tag{20}$$

The equation of motion is

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\mu^2 x. \tag{21}$$

This can be integrated to

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sqrt{2\left(E - \frac{1}{2}\mu^2 x^2\right)}. \tag{22}$$

The solution of the above equation is

$$x(t, \phi) = \frac{\sqrt{2E}}{\mu} \sin(\mu t + \phi) \tag{23}$$

where ϕ is an arbitrary constant which is determined by the initial conditions of the trajectory. Figure 1 shows a trajectory in the complex x-plane which starts from the turning point $\frac{\sqrt{2E}}{\mu}$. The real and imaginary parts x_r and x_i of x are given by

$$x_\mathrm{r} = \sin(\mu_\mathrm{r} t + \phi) \cosh(\mu_\mathrm{i} t) \tag{24}$$

$$x_\mathrm{i} = \cos(\mu_\mathrm{r} t + \phi) \sinh(\mu_\mathrm{i} t). \tag{25}$$

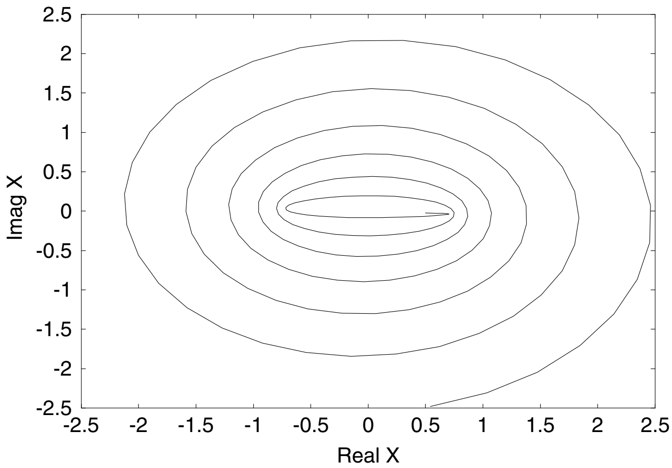

Figure 1. Trajectory for the potential $v(x) = \frac{1}{2}\mu^2 x^2$ in the complex x-plane which starts from the turning point $\frac{\sqrt{2E}}{\mu}$. $E = 1.0$ and $\mu = 1 + \mathrm{i}$.

Note that when $\mu_\mathrm{i} = 0$, i.e. for real frequencies, trajectories that start from turning points (ϕ is real) are straight lines joining two turning points while all the trajectories starting outside the real axis (ϕ is complex) or outside these two turning points are ellipses encircling two turning points. The period of the motion is $\frac{2\pi}{\mu}$. When $\mu_\mathrm{r} = 0$ and $\mu_\mathrm{i} \neq 0$, trajectories exponentially diverge to ∞ as $t \to \infty$. When $\mu_\mathrm{r} \neq 0$ and $\mu_\mathrm{i} \neq 0$,

$$x(t, \phi) \sim -\mathrm{sign}(\mu_\mathrm{i}) \frac{\mathrm{e}^{|\mu_\mathrm{i}|t}}{2\mathrm{i}} \, \mathrm{e}^{-\mathrm{i}(\mu_\mathrm{r} t + \phi)\mathrm{sign}(\mu_\mathrm{i})} \tag{26}$$

and trajectories spiral around as shown in figure 1. However, they infinitely oscillate as evident from equations (24) and (25) as $t \to \infty$.

Now we calculate the Lyapunov exponent for an arbitrary trajectory of this system. For this system

$$\frac{\Delta x(t)}{\Delta x(0)} = \frac{\cos(\mu t + \phi)}{\cos(\phi)} \tag{27}$$

$$\frac{\Delta p(t)}{\Delta p(0)} = \frac{\sin(\mu t + \phi)}{\sin(\phi)}. \tag{28}$$

For large t

$$\cos(\mu t + \phi) \sim \frac{\mathrm{e}^{|\mu_\mathrm{i}|t}}{2} \, \mathrm{e}^{-\mathrm{i}(\mu_\mathrm{r} t + \phi) \, \mathrm{sign} \, (\mu_\mathrm{i})} \tag{29}$$

$$\sin(\mu t + \phi) \sim -\mathrm{sign}(\mu_\mathrm{i}) \frac{\mathrm{e}^{|\mu_\mathrm{i}|t}}{2\mathrm{i}} \, \mathrm{e}^{-\mathrm{i}(\mu_\mathrm{r} t + \phi) \, \mathrm{sign} \, (\mu_\mathrm{i})}. \tag{30}$$

As $t \to \infty$

$$\left| \frac{\Delta x(t)}{\Delta x(0)} \right| \sim \frac{\mathrm{e}^{|\mu_\mathrm{i}|t}}{2|\cos(\phi)|} \tag{31}$$

$$\left| \frac{\Delta p(t)}{\Delta p(0)} \right| \sim \frac{\mathrm{e}^{|\mu_\mathrm{i}|t}}{2|\sin(\phi)|}. \tag{32}$$

Therefore Lyapunov exponent λ is

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \ln \left\{ \left| \frac{\Delta x(t)}{\Delta x(0)} \right| + \left| \frac{\Delta p(t)}{\Delta p(0)} \right| \right\} \tag{33}$$

$$= |\mu_{\mathrm{i}}|. \tag{34}$$

When the frequencies are real, i.e. $\mu_{\mathrm{i}} = 0$, and motion is non-chaotic as expected. When $\mu_{\mathrm{i}} \neq 0$ Lyapunov exponent is positive and hence motion is classified as chaotic.

4. Classical motion and Lyapunov exponents of the complex cubic anharmonic oscillator

In this section first we study in detail the classical motion of the complex cubic anharmonic oscillator. We assume that the Hamiltonian has the form

$$H = \frac{p^2}{2} + \mu x^3 \tag{35}$$

where $\mu = \mu_{\mathrm{r}}\,\mathrm{e}^{\mathrm{i}\theta}$ is a complex constant. Note that when $\theta = \frac{\pi}{2}$ Hamiltonian is PT symmetric.

The classical trajectories of the PT symmetric case have been studied by Bender *et al* in detail [21]. They have calculated the periods of oscillatory motion and escape times of the trajectories which reach infinity in finite time. They raised the question '*does the breaking of P and T symmetry allow for unbounded chaotic solution*s?'

The equation of motion is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = p = \sqrt{E - \mu x^3}. \tag{36}$$

Now we rescale x and t by real numbers such that the above equation has the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = p = \sqrt{1 - \mathrm{e}^{\mathrm{i}\theta} x^3}. \tag{37}$$

The turning points of this system are

$$x_0 = \mathrm{e}^{\mathrm{i}\frac{(2\pi-\theta)}{3}} \tag{38}$$

$$x_1 = \mathrm{e}^{-\mathrm{i}\frac{\theta}{3}} \tag{39}$$

$$x_2 = \mathrm{e}^{-\mathrm{i}\frac{(2\pi+\theta)}{3}}. \tag{40}$$

Now we write $1 - \mathrm{e}^{\mathrm{i}\theta} x^3 = \mathrm{e}^{\mathrm{i}\theta}(x_0 - x)(x_1 - x)(x_2 - x)$ and integrate equation (37). Then we have

$$\int \frac{\mathrm{d}x}{\sqrt{(x_0 - x)(x_1 - x)(x_2 - x)}} = \left(\mathrm{e}^{\mathrm{i}\frac{\theta}{2}}\right) t + c \tag{41}$$

where c is the constant of integration which depends on initial conditions. The left-hand side of the above equation is an elliptic integral of the first kind and hence equation (41) becomes

$$\frac{1}{\sqrt{x_0 - x_1}} F\left(\sin^{-1}\left[\sqrt{\frac{x - x_0}{x_2 - x_0}} \right], \frac{x_2 - x_0}{x_1 - x_0} \right) = \left(\mathrm{e}^{\mathrm{i}\frac{\theta}{2}}\right) t + c \tag{42}$$

where F is an elliptic function. We invert the above equation in terms of Jacobian elliptic function sn [24] as

$$x(t) = x_0 + (x_2 - x_0)\,\mathrm{sn}^2\left[\frac{\left(\mathrm{e}^{\mathrm{i}\frac{\theta}{2}}\right)\sqrt{x_0 - x_1}}{2} t + \alpha, \kappa^2 \right] \tag{43}$$

where modulus $\kappa = \left(\frac{x_2 - x_0}{x_1 - x_0}\right)^{1/2}$ and α is an arbitrary constant which is determined by the initial conditions. Also note that $x(t)$ in the above equation is still a solution of (37), when x_0, x_1

and x_2 are cyclically changed (e.g. $x_2 \to x_0 \to x_1 \to x_2$). In order to understand how the trajectories behave, we need to recognize the periodic, bounded and unbounded properties of the function $x(t)$. First we find complementary modulus κ' and complete elliptic functions K and K'. They are defined [24] by

$$\kappa'^2 = 1 - \kappa^2 = \left(\frac{x_1 - x_2}{x_1 - x_0}\right) \tag{44}$$

$$K = \int_0^{\frac{\pi}{2}} (1 - k^2 \sin^2(\phi))^{-\frac{1}{2}}\, \mathrm{d}\phi \tag{45}$$

$$K' = \int_0^1 (1 - t^2)^{-\frac{1}{2}} (1 - \kappa'^2 t^2)^{-\frac{1}{2}}\, \mathrm{d}t. \tag{46}$$

K and K' are evaluated directly from the above equations and they are independent of phase angle θ. After some simplifications

$$\kappa = \left[1 + \exp\left(-\mathrm{i}\frac{2\pi}{3}\right)\right]^{1/2} \tag{47}$$

$$\kappa' = -\exp\left(-\mathrm{i}\frac{2\pi}{6}\right) \tag{48}$$

$$K = \frac{\mathrm{e}^{-\mathrm{i}\frac{\pi}{12}}\sqrt{\pi}\,\Gamma\left(\frac{1}{6}\right)}{2(3^{3/4})\Gamma\left(\frac{2}{3}\right)} \tag{49}$$

$$K' = \frac{\mathrm{e}^{\mathrm{i}\frac{\pi}{12}}\sqrt{\pi}\,\Gamma\left(\frac{1}{6}\right)}{2(3^{3/4})\Gamma\left(\frac{2}{3}\right)}. \tag{50}$$

Before calculating Lyapunov exponents of this system, we study the phase space trajectories in detail. With the above quantities, we determine the periodic and unbounded nature of the trajectory of the Hamiltonian (35).

Since α in (43) alone determines the trajectory, depending on the value of α, trajectories become bounded or unbounded. Jacobian elliptic function $\mathrm{sn}(u)$ is a doubly periodic function of u with periods $4K$ and $2\mathrm{i}K'$. It is analytic except at the points congruent to $\mathrm{i}K'$ or to $2K + \mathrm{i}K' (\mathrm{mod}\ 4K, 2\mathrm{i}K')$. These points are simple poles. For more information on properties of Jacobian elliptic functions see [24]. Now we relate the periodicity of Jacobian elliptic function $\mathrm{sn}(u)$ to the periodic motion of trajectories and poles of $\mathrm{sn}(u)$ to the unbounded motion as follows:

The trajectory becomes unbounded and the particle escapes to infinity when conditions

$$\frac{\mathrm{e}^{\mathrm{i}\frac{\theta}{2}}\sqrt{x_0 - x_1}}{2} t + \alpha = \mathrm{i}K' \tag{51}$$

or

$$\frac{\mathrm{e}^{\mathrm{i}\frac{\theta}{2}}\sqrt{x_0 - x_1}}{2} t + \alpha = 2K + \mathrm{i}K' \tag{52}$$

are satisfied for some real positive t and the time taken for the particle to escape to ∞ is given by

$$T = \frac{2(\mathrm{i}K' - \alpha)\,\mathrm{e}^{-\mathrm{i}\frac{\theta}{2}}}{\sqrt{x_0 - x_1}} \tag{53}$$

or

$$T = \frac{2(2K + \mathrm{i}K' - \alpha)\,\mathrm{e}^{-\mathrm{i}\frac{\theta}{2}}}{\sqrt{x_0 - x_1}} \tag{54}$$

depending on which equation of (51) or (52) is satisfied. If equations (51) and (52) do not have real solutions, the particle will not escape to ∞ in a finite time. On the other hand, if the particle does not escape to ∞ and the trajectory satisfies the following equation for real positive t and for some integer n and m:

$$\frac{e^{i\frac{\theta}{2}}\sqrt{x_0 - x_1}}{2}t + \alpha = 4nK + 2miK' \tag{55}$$

then the trajectory is periodic with the period

$$T = \frac{4(2n_0K + m_0 iK')\, e^{-i\frac{\theta}{2}}}{\sqrt{x_0 - x_1}} \tag{56}$$

where n_0 and m_0 are the smallest n and m which satisfy equation (53) for real positive t. Now we illustrate the above ideas by considering the case when $\theta = \frac{\pi}{2}$. For this θ the Hamiltonian in (35) is PT symmetric and the turning points are $x_0 = i$, $x_1 = \frac{\sqrt{3}}{2} - \frac{i}{2}$, $x_2 = -\frac{\sqrt{3}}{2} - \frac{i}{2}$. The trajectories are given by

$$x(t) = i - \sqrt{3}\, e^{i\frac{\pi}{3}}\, \mathrm{sn}^2\left[\frac{\left(e^{i\frac{7\pi}{12}}\right)3^{\frac{1}{4}}}{2}t + \alpha, \kappa^2\right] \tag{57}$$

and α determines the initial conditions of the trajectories. For this Hamiltonian system, two types of motion were observed; unbounded and periodic. First we consider the trajectory starting from the turning point $x_0 = i$. For this trajectory $\alpha = 0$ and

$$x(t) = i - \sqrt{3}\, e^{i\frac{\pi}{3}}\, \mathrm{sn}^2\left[\frac{\left(e^{i\frac{7\pi}{12}}\right)3^{\frac{1}{4}}}{2}t, \kappa^2\right]. \tag{58}$$

Since

$$iK' = \frac{e^{i\frac{7\pi}{12}}\sqrt{\pi}\,\Gamma\left(\frac{1}{6}\right)}{2(3^{3/4})\Gamma\left(\frac{2}{3}\right)}$$

this trajectory is unbounded and goes to $i\infty$ as t reaches $\frac{\sqrt{\pi}\Gamma(\frac{1}{6})}{3\Gamma(\frac{2}{3})}$.

Now we consider a trajectory which starts from any point in the positive imaginary axis above the turning point x_0. That is $x(0) = ai$. With real number $a > 1$, α for this trajectory is given by

$$\alpha = F\left(\sin^{-1}\sqrt{\frac{a-1}{2\sin\left(\frac{2\pi}{3}\right)}}\, e^{i\frac{5\pi}{12}}, k^2\right) \tag{59}$$

where F is the elliptic function of the first kind. It was found that when $x(0) = ai$ and α is real,

$$F\left(\sin^{-1}\sqrt{\frac{a-1}{2\sin\left(\frac{2\pi}{3}\right)}}\, e^{i\frac{5\pi}{12}}, k^2\right) = \alpha_0\, e^{i\frac{7\pi}{12}}$$

where α_0 is a real number. Hence

$$t = \left(\frac{2(iK' - \alpha)}{3^{\frac{1}{4}}}\, e^{-i\frac{7\pi}{12}}\right)$$

is real and therefore all the trajectories starting from points on the positive imaginary axis above i will reach $i\infty$ in a finite time

$$t = \frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{6}\right)}{3\Gamma\left(\frac{2}{3}\right)} - \frac{2\alpha_0}{3^{\frac{1}{4}}}. \tag{60}$$

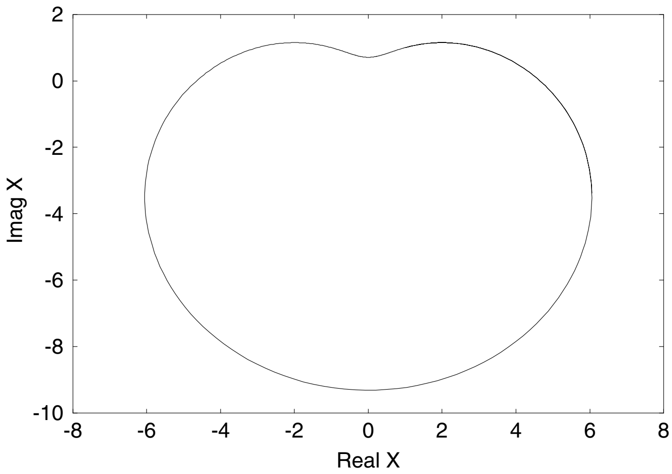

Figure 2. A typical periodic trajectory of the potential $v(x) = \mathrm{i}x^3$ which starts from outside the imaginary axis. This has the period $T = \frac{2}{3}\frac{\sqrt{\pi}\Gamma(\frac{1}{6})}{\Gamma(\frac{2}{3})}$.

These are the only trajectories which are unbounded and go to $\mathrm{i}\infty$ in finite time. Now we consider a trajectory starting from the turning point x_1. It is convenient to write $x(t)$ as

$$x(t) = x_1 + (x_0 - x_1)\,\mathrm{sn}^2\left[\frac{\left(\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\right)\sqrt{x_1 - x_2}}{2}t, \kappa^2\right] \tag{61}$$

$$= x_1 - \sqrt{3}\mathrm{e}^{-\mathrm{i}\frac{\pi}{3}}\,\mathrm{sn}^2\left[\frac{\left(\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\right)\sqrt{3}}{2}t, \kappa^2\right]. \tag{62}$$

Obviously $x(t)$ has no poles for all real t values. However, this is a periodic function with the period T satisfying $\frac{(\mathrm{e}^{\mathrm{i}\frac{\pi}{4}})3^{\frac{1}{4}}}{2}T = 4(K + \mathrm{i}K')$.

Hence the trajectory is periodic with the period $T = \frac{2}{3}\frac{\sqrt{\pi}\Gamma(\frac{1}{6})}{\Gamma(\frac{2}{3})}$. Similar to the arguments used above, it can be shown that any trajectory which is not starting from the positive imaginary axis above i is periodic and has the period $T = \frac{2}{3}\frac{\sqrt{\pi}\Gamma(\frac{1}{6})}{\Gamma(\frac{2}{3})}$ (see figure 2).

Now we consider the general case where Hamiltonian is not necessarily PT symmetric. That is θ can have any value between 0 and 2π. For given θ there are two types of motions found as in the PT-symmetric case. The first type of motion is the particle escapes to infinity in finite time. In the PT-symmetric case $\left(\theta = \frac{\pi}{2}\right)$, we observed that the particle escapes to infinity if it starts from a point on the imaginary axis above i. For the other values of θ, the particle escapes to infinity if it starts from points along a specific curve. These curves, for some θ values, are shown in figure 3. Another interesting observation is that when a trajectory starts from any point on these curves, it traverses along the same curve to infinity in a finite time.

The second type of trajectories are non-periodic but oscillatory as shown in figure 4. These trajectories do not close themselves in finite time. Trajectories go around each and every turning point and then move away but they return to the neighbourhood of turning points. This takes place indefinitely.

In order to find trajectories which go to infinity in finite time, we use the condition (51) derived earlier. After some simplifications equation (51) becomes

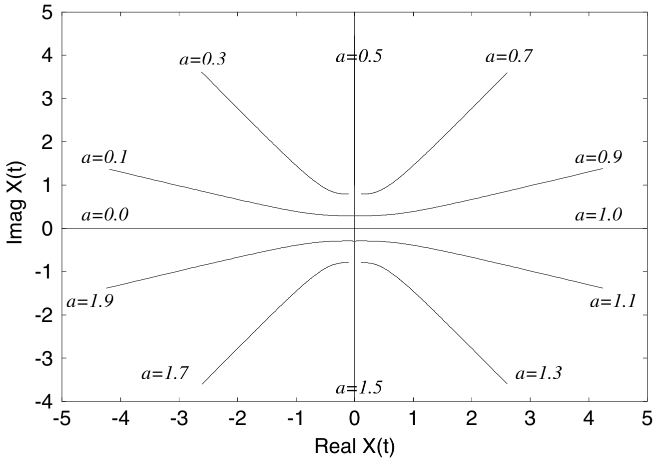

Figure 3. When μ is complex, trajectories of the cubic potential $v(x) = \mu x^3$, go to infinity if they start from points along certain curves. Figure shows twelve such curves for twelve values of θ. Each curve is unique for a given θ.

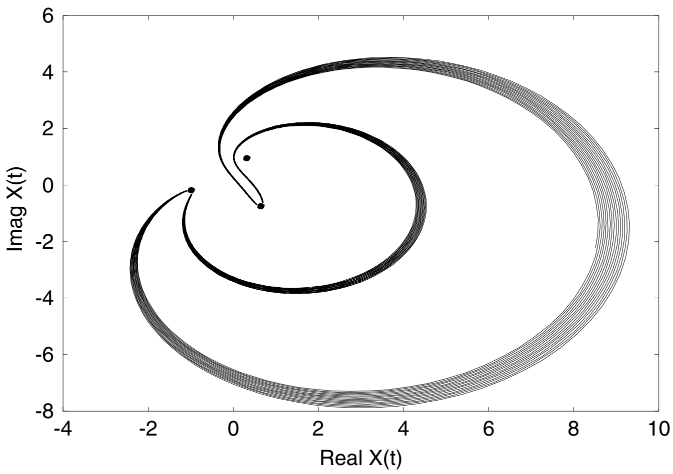

Figure 4. A typical trajectory of the cubic potential $v(x) = \mu x^3$ which does not go to infinity in finite time, when μ is complex. These kinds of trajectories go around each and every turning point and then move away, but they will come back to the neighbourhood of turning points again. This takes place indefinitely.

$$t = \frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{6}\right)}{3\Gamma\left(\frac{2}{3}\right)}\,\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}\,\mathrm{e}^{-\mathrm{i}\frac{\theta}{3}} - \frac{2\alpha(x)}{3^{\frac{1}{4}}}\,\mathrm{e}^{-\mathrm{i}\frac{5\pi}{12}}\,\mathrm{e}^{-\mathrm{i}\frac{\theta}{3}} \tag{63}$$

where $\alpha(x)$ is given by

$$\alpha(x) = F[\sin^{-1}\sqrt{Q}, k] \tag{64}$$

and

$$Q = \frac{x - x_0}{x_2 - x_0} = \frac{1}{2}\,\frac{\left(x\,\mathrm{e}^{\mathrm{i}\frac{\theta}{3}}\,\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}} - \mathrm{e}^{\mathrm{i}\frac{\pi}{6}}\right)}{\sin\left(\frac{2\pi}{3}\right)}. \tag{65}$$

$F[\sin^{-1}\sqrt{Q}, k]$ is an elliptic function of the first kind.

Table 1. For various values of θ (where $\theta = a\pi$), initial values $r(0)_{\min}$ and $\phi(0)$ of the trajectories which escape to infinity in finite time are shown. The angles $\phi(0)$ are given in radians.

$x = r(t)\,\mathrm{e}^{\mathrm{i}\varphi(t)}$, $\theta = a\pi$		
a	$r(0)$ min	$\varphi(0)$
0	0	$\frac{\pi}{2}$
0.1	0.2929	1.562 39
0.3	0.8058	1.691 02
0.5	1.0000	1.570 7
0.7	0.8058	1.443 7
0.9	0.2929	1.558 9
1.1	0.2929	4.724 3
1.3	0.8058	4.839 4
1.5	1.0000	4.712 3
1.7	0.8058	4.585 3
1.9	0.2929	4.700 42
2.0	0	$\frac{3\pi}{2}$

A trajectory which starts from a point x in the complex plane will go to infinity in a finite time if x satisfies the following equation:

$$\mathrm{Im}\left[\left(\frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{6}\right)}{3\Gamma\left(\frac{2}{3}\right)}\,\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}} - \frac{2\alpha(x)}{3^{\frac{1}{4}}}\,\mathrm{e}^{-\mathrm{i}\frac{5\pi}{12}}\right)\mathrm{e}^{-\mathrm{i}\frac{\theta}{3}}\right] = 0. \tag{66}$$

On the other hand every trajectory starting from points on the curves in figure 4 goes to infinity along the same curve in finite time T_∞ where

$$T_\infty = \left(\frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{6}\right)}{3\Gamma\left(\frac{2}{3}\right)}\,\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}} - \frac{2\alpha(x)}{3^{\frac{1}{4}}}\,\mathrm{e}^{-\mathrm{i}\frac{5\pi}{12}}\right)\mathrm{e}^{-\mathrm{i}\frac{\theta}{3}}. \tag{67}$$

We solved equation (66) for various values of θ. Table 1 shows the minimum values of $|x|$ for which the particle escapes to infinity in finite time.

In order to find all the periodic trajectories of this system for arbitrary θ, we use (56) and demand that T should be real in (56). For a given x one can obtain θ for which the trajectory starting from x is periodic. The condition which should be satisfied for this situation is

$$\mathrm{Im}\left[\frac{4\sqrt{\pi}\,\Gamma\left(\frac{1}{6}\right)}{3\Gamma\left(\frac{2}{3}\right)}\,\mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}}\,\mathrm{e}^{-\mathrm{i}\frac{2\theta}{3}}\right] = 0 \tag{68}$$

or

$$\mathrm{Im}\left[\frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{6}\right)}{6\Gamma\left(\frac{2}{3}\right)}\,\mathrm{e}^{-\mathrm{i}\frac{5\pi}{12}}\,\mathrm{e}^{-\mathrm{i}\frac{\theta}{3}}\,\mathrm{e}^{\mathrm{i}\varphi}\right] = 0 \tag{69}$$

where

$$\varphi = \tan^{-1}\left[\frac{4n\sin\left[\frac{\pi}{12}\right] - 2m\cos\left[\frac{\pi}{12}\right]}{2m\sin\left[\frac{\pi}{12}\right] - 4n\cos\left[\frac{\pi}{12}\right]}\right].$$

Note that the above two equations are independent of x. Therefore from (67), we obtain $\theta = \pi \pm \frac{3}{2}l\pi$ where $l = 0, 1, 2, 3, \ldots$ satisfies the above conditions regardless of the starting point of the trajectory. For these θ values all the trajectories are periodic with a real period

$$T = \frac{4\sqrt{\pi}\,\Gamma\left(\frac{1}{6}\right)}{3\Gamma\left(\frac{2}{3}\right)}\,\mathrm{e}^{-\mathrm{i}\frac{2(\pi+\theta)}{3}}$$

unless they satisfy (66) with $t < T$, where t is given by (63). Similarly from (68), expressions for θ and T can be obtained as $\theta = 3\varphi - \frac{5\pi}{4} \pm 3l\pi$, where $l = 0, 1, 2, 3, \ldots$ and $T = \frac{\sqrt{\pi}\Gamma(\frac{1}{6})}{6\Gamma(\frac{2}{3})} \mathrm{e}^{-\mathrm{i}\frac{5\pi}{12}} \mathrm{e}^{-\mathrm{i}\frac{\theta}{3}} \mathrm{e}^{\mathrm{i}\varphi}$. Note that n and m in the expression for φ above are the minimum integer values which satisfy equation (68).

Now we calculate Lyapunov exponents of this system. As before two cases are considered separately. One is the PT symmetric case, where $\theta = \frac{\pi}{2}$, and the other is the non-PT symmetric case , where $\theta \neq \frac{\pi}{2}$. For the PT symmetric case $x(t)$ is given by

$$x(t) = G(t, \alpha) = \mathrm{i} - \sqrt{3}\mathrm{e}^{\mathrm{i}\frac{\pi}{3}} \,\mathrm{sn}^2 \left[\frac{\left(\mathrm{e}^{\mathrm{i}\frac{7\pi}{12}}\right) 3^{\frac{1}{4}}}{2} t + \alpha, \kappa^2 \right]. \tag{70}$$

Since in the PT symmetric case all the trajectories are periodic unless they start from pure imaginary $x(0)$ above i. Let $x(0) = a\mathrm{i}$ where a is a real number greater than or equal to unity. Since the particle escapes to infinity as $t \to T_\infty = \frac{\sqrt{\pi}\Gamma(\frac{1}{6})}{3\Gamma(\frac{2}{3})} - \frac{2\alpha_0}{3^{\frac{1}{4}}}$ where

$$\alpha_0 = \mathrm{e}^{-\mathrm{i}\frac{7\pi}{12}} F \left(\sin^{-1} \sqrt{\frac{a-1}{2\sin\left(\frac{2\pi}{3}\right)}}\, \mathrm{e}^{\mathrm{i}\frac{5\pi}{12}}, k^2 \right)$$

we rewrite the equation (17) as

$$\lambda = \lim_{t - T_\infty} \frac{1}{t} \ln \left[\left| \frac{\frac{\partial G}{\partial \alpha}}{\frac{\partial G}{\partial \alpha}(t=0)} \right| + \left| \frac{\frac{\partial^2 G}{\partial t \partial \alpha}}{\frac{\partial^2 G}{\partial t \partial \alpha}(t=0)} \right| \right]. \tag{71}$$

By using relations

$$\frac{\mathrm{d}(\mathrm{sn}(u))}{\mathrm{d}u} = \mathrm{cn}(u)\,\mathrm{dn}(u) \tag{72}$$

$$\frac{\mathrm{d}(\mathrm{cn}(u))}{\mathrm{d}u} = -\mathrm{sn}(u)\,\mathrm{dn}(u) \tag{73}$$

$$\frac{\mathrm{d}(\mathrm{dn}(u))}{\mathrm{d}u} = -k\,\mathrm{sn}(u)\,\mathrm{cn}(u) \tag{74}$$

we obtain

$$\frac{\frac{\partial G}{\partial \alpha}}{\frac{\partial G}{\partial \alpha}(t=0)} = \frac{\mathrm{sn}(u)\,\mathrm{cn}(u)\,\mathrm{dn}(u)}{\mathrm{sn}(\alpha)\,\mathrm{cn}(\alpha)\,\mathrm{dn}(\alpha)} \tag{75}$$

$$\frac{\frac{\partial^2 G}{\partial t \partial \alpha}}{\frac{\partial^2 G}{\partial t \partial \alpha}(t=0)} = \frac{\mathrm{cn}^2(u)\,\mathrm{dn}^2(u) - \mathrm{sn}^2(u)\,\mathrm{dn}^2(u) - k\,\mathrm{sn}^2(u)\,\mathrm{cn}^2(u)}{\mathrm{cn}^2(\alpha)\,\mathrm{dn}^2(\alpha) - \mathrm{sn}^2(\alpha)\,\mathrm{dn}^2(\alpha) - k\,\mathrm{sn}^2(\alpha)\,\mathrm{cn}^2(\alpha)} \tag{76}$$

where $u = \frac{\left(\mathrm{e}^{\mathrm{i}\frac{7\pi}{12}}\right)3^{\frac{1}{4}}}{2} t + \alpha$. Note that when $u \to \mathrm{i}K'$, $\mathrm{sn}(u) \to \infty$, $\mathrm{cn}(u) \to \infty$ and $\mathrm{dn}(u) \to \infty$. Now we expand $\mathrm{sn}(u)$, $\mathrm{cn}(u)$, $\mathrm{dn}(u)$ near $\mathrm{i}K'$ as

$$\mathrm{sn}(\mathrm{i}K' + \Delta u) = \frac{1}{\sqrt{k}\Delta u} + \frac{1+k}{6\sqrt{k}}\Delta u + \cdots \tag{77}$$

$$\mathrm{cn}(\mathrm{i}K' + \Delta u) = \frac{-\mathrm{i}}{\sqrt{k}\Delta u} + \frac{2k-1}{6\sqrt{k}}\mathrm{i}\Delta u + \cdots \tag{78}$$

$$\mathrm{dn}(\mathrm{i}K' + \Delta u) = \frac{-\mathrm{i}}{\Delta u} + \frac{2-k}{6\sqrt{k}}\mathrm{i}\Delta u + \cdots \tag{79}$$

we approximate $\mathrm{sn}(u)$, $\mathrm{cn}(u)$, and $\mathrm{dn}(u)$

$$[\mathrm{sn}(\mathrm{i}K' + \Delta u)\,\mathrm{cn}(\mathrm{i}K' + \Delta u)\,\mathrm{dn}(\mathrm{i}K' + \Delta u)] \to -\frac{1}{k\Delta u^3} \tag{80}$$

and

$$[\mathrm{cn}^2(u)\,\mathrm{dn}^2(u) - \mathrm{sn}^2(u)\,\mathrm{dn}^2(u) - k\,\mathrm{sn}^2(u)\,\mathrm{cn}^2(u)] \to -\frac{3}{k\Delta u^4}. \tag{81}$$

Now we have

$$\lambda = \lim_{\Delta u \to 0} \Delta u \ln\left[\left|\frac{1}{k\Delta u^3}\right| + \left|\frac{1}{k\Delta u^4}\right|\right] \tag{82}$$

and $\lambda = 0$. This shows neighbouring trajectories do not diverge exponentially. However, neighbouring trajectories show power-law divergence. For the non-PT symmetric case $x(t)$ is given by

$$x(t) = x_0 + (x_2 - x_0)\,\mathrm{sn}^2\left[\frac{\left(\mathrm{e}^{\mathrm{i}\frac{\theta}{2}}\right)\sqrt{x_0 - x_1}}{2}t + \alpha, \kappa^2\right] \tag{83}$$

where α is written as

$$\alpha = F\left(\sin^{-1}\left[\sqrt{\frac{x(0) - x_0}{x_2 - x_0}}\right], k\right) \tag{84}$$

F is an elliptic function of the first kind. Following a similar procedure as in the case of PT symmetry, we obtain the same expression for Lyapunov exponent λ for the trajectories which go to infinity in finite time T_∞. That is

$$\lambda = \lim_{\Delta u \to 0} \Delta u \ln\left[\left|\frac{1}{k\Delta u^3}\right| + \left|\frac{1}{k\Delta u^4}\right|\right] \tag{85}$$

and hence $\lambda = 0$. We conclude that neighbouring trajectories in this case also do not diverge exponentially but diverge as the third power. In all the other cases neighbouring trajectories do not diverge and Lyapunov exponent λ is zero.

5. Summary and discussion

In this paper we studied the classical motion of two 1D non-Hermitian Hamiltonian systems, the complex harmonic oscillator and the complex cubic potential analytically. In addition to the PT symmetric case, *non*-pseudo Hermitian 1D cases were studied. These 1D potentials can be combined to form 2D separable pseudo-Hermitian Hamiltonians which have the classical motion of a 1D non-pseudo Hermitian complex system. This can be explained with the following example. Consider the Hamiltonian $H_1 = \frac{p^2}{2} + \mu x^3$ with $\mu = \mu_\mathrm{r} + \mathrm{i}\mu_\mathrm{i}$. When $\mu_\mathrm{r} \neq 0$ and $\mu_\mathrm{i} \neq 0$, H_1 is neither PT symmetric nor pseudo-Hermitian. However, the 2D Hamiltonian H can be constructed as $H = \frac{p_x^2}{2} + \frac{p_y^2}{2} + \mu x^3 + \overline{\mu} y^3$ where $\overline{\mu}$ is the complex conjugate of μ. This 2D Hamiltonian is pseudo-Hermitian under the exchange operator; $\eta : x \leftrightarrow y,\ p_x \leftrightarrow p_y$. Quantum mechanical eigenvalues of H are either real or come as conjugate pairs. Since H is separable, classical motion is still governed by 1D complex Hamiltonians $H_1 = \frac{p_x^2}{2} + \mu x^3$ and $H_2 = \frac{p_y^2}{2} + \overline{\mu} y^3$. We studied analytically the classical motion of these cubic systems in detail. Also we derived the Lyapunov exponent for the general 1D Hamiltonian of the form $H = \frac{p^2}{2} + V(x)$ and the same was obtained explicitly for the potentials $V(x) = \mu x^2$ and $V(x) = \mu x^3$. It is important to mention that the Lyapunov exponents of escape trajectories of this potential cannot be calculated numerically as the particle escapes to infinity very rapidly. We investigated whether there is any connection between PT symmetry and non-chaotic motion of classical trajectories and found that when $H_1 = \frac{p_x^2}{2} + \mu x^3$ is PT symmetric, certain neighbouring trajectories diverge not exponentially but cubically.

One of the interesting questions to answer is that whether there is a connection between reality of the entire quantum spectrum and regularity of the classical trajectories of the 1D pseudo-Hermitian systems. As is well known in 2D and higher dimensions, real Hermitian Hamiltonians can have classical chaos (e.g. Henon Heiles system). However, in 1D Hermitian Hamiltonians cannot have chaos. But as shown in this paper, 1D complex non-Hermitian systems can have chaos in complex phase space. The two systems which have been investigated in this paper support the idea that when the quantum eigenspectra are entirely real, the classical trajectories are regular and when the eigenvalues are entirely complex, all the trajectories are chaotic.

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